12.12 Calculus I Review

BACKGROUND: In this lab, you will review the domain of a function, composition of functions, differentiation (using the chain rule), and integration (to compute area), as well as the Maple commands for these operations.

ASSIGNMENT: In problems 1-9, consider the following four functions:

$$f(x) = x^2, \quad g(x) = \sqrt{4-x}, \quad h(x) = (f \circ g)(x), \quad k(x) = (g \circ f)(x)$$

1. Enter f and g into Maple using arrow definitions, e.g.

   ```
   >  f:=x->x^2;
   ```

2. Enter h and k into Maple using Maple's composition symbol, `@`. Compute Maple's expressions for $h(x)$ and $k(x)$, e.g.

   ```
   >  h:=f@g;    h(x);
   ```

3. Using Maple's `plot` command, graph f and g for $-6 \leq x \leq 6$ in one plot. Make f red and thin. Make g blue and thick.

4. The function f has all real numbers for its domain. What is the domain for the function g? Explain why the plot for g is not drawn for all numbers between -6 and 6.

5. Graph h and k for $-6 \leq x \leq 6$ in one plot. Make h red and thin. Make k blue and thick.

6. Explain why the straight line $y = 4 - x$ should not be the graph of h, and draw a better graph.

7. Compute the derivatives of $h(x)$ and $k(x)$. Are Maple's answers correct? Why?
   NOTE: You can have Maple calculate derivatives with either the `D` operator or the `diff` operator.

   ```
   >  deriv_h:=D(h)(x);
   >  deriv_h:=diff(h(x),x);
   ```

8. Sketch the region in the first quadrant bounded by the curves $y = 0$, $y = f(x)$ and $y = g(x)$.

9. Calculate the area of the region you sketched in problem 8, by using the Maple's `Int` and `value` commands
   NOTE: Instead of using Maple's `Int` and `value` commands, you could use `int`. If you do this Maple will not print out the integrals; only the answer will be printed. *Get in the habit of using commands that print out what you want calculated.* This makes it a lot easier to find and correct typos and other mistakes.

10. Explain the differences in output between the following approaches to calculating the derivative of the function $\dfrac{1}{x(x+2)}$.

(a) > `diff(1/x*(x+2),x);`

(b) > `p:=1/x*(x+2); diff(p,x);`

(c) > `p:=(x*(x+2))^(-1); diff(p,x);`

(d) > `Diff(1/x*(x+2),x); value(%);`

Which values are correct? How can the others be fixed? After fixing them, which approach do you feel is best? Why?

12.13 Calibrating a Dipstick

BACKGROUND: Review volumes of revolution.

ASSIGNMENT:

1. Suppose the quarter-circle $y = -\sqrt{64 - x^2}$ for $0 \le x \le 8$ is rotated about the y-axis to create a hemispherical bowl (8 units deep). Construct a dipstick which when inserted vertically into the bowl will determine when the bowl is one quarter full, one-half full and three-quarters full. Plot the percent full as a function of the height on the dipstick. Then plot the height on the dipstick as a function of the percent full.

   HINT: Recall the discussion of plotting inverse functions in Section 3.3.
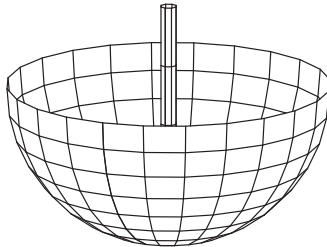
2. Now repeat the same problem assuming that the bowl is generated by rotating the curve $y = -(256 - x^4)^{1/4}$ for $0 \le x \le 4$ around the y-axis.

12.14   The Area of a Unit p-Ball

GOAL: In this project, you will determine the area of a unit p-ball in the plane for different values of p and look at their limiting characteristics.

DEFINITIONS: The p-norm of a vector $\vec{v} = (x, y)$ is $|\vec{v}|_p = \sqrt[p]{|x|^p + |y|^p}$ instead of the standard Euclidean 2-norm $|\vec{v}|_2 = \sqrt{|x|^2 + |y|^2}$. So a p-normed circle of radius R with center at the origin is the set of points (x, y) satisfying

$$|x|^p + |y|^p = R^p,$$

and a p-ball is the interior of a p-normed circle. So you need to compute the area of the region satisfying $|x|^p + |y|^p \leq 1$.

1. Using `implicitplot` or just `plot` with `scaling=constrained`, graph several unit p-circles in the plane with $p \geq 1$. Specifically, superimpose the curves $|x|^p + |y|^p = 1$ for $p = 1,\ 2,\ 3,\ 4,\ 5$. Notice they are convex.

2. Make a conjecture as to the limiting shape and area of these p-balls as $p \to \infty$.

3. Using `implicitplot` or just `plot` with `scaling=constrained`, graph several unit p-circles in the plane with $0 < p \leq 1$. Specifically, superimpose the curves $|x|^p + |y|^p = 1$ for $p = 1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5}$. Notice they are concave for $p < 1$.

4. Make a conjecture as to the limiting "shape" and "area" of these p-balls as $p \to 0^+$.

5. For $p = 1,\ 2,\ 3,\ 4,\ 5$, compute the area of the unit p-ball $|x|^p + |y|^p \leq 1$.

   HINT: For each value of p, the fact that the p-ball is symmetric with respect to both the x-axis and the y-axis means that the total area is 4 times the area of the part of the p-ball in the first quadrant.

6. Obtain a general formula for the area of the unit p-ball for $p \geq 1$.

7. What is the limiting value of the area of the unit p-ball as $p \to \infty$? Apply Maple's `Limit` and `value` commands to either the integral or your general formula. Does this agree with your conjecture?

8. For $p = \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5}$, compute the area of the unit p-ball $|x|^p + |y|^p \leq 1$.

9. Obtain a general formula for the area of the unit p-ball for $0 < p < 1$.

   HINTS: Let $p = \dfrac{1}{q}$. The formulas in #6 and #9 are the same.

10. What is the limiting value of the area of the unit p-ball as $p \to 0^+$? Does this agree with your conjecture?

12.15   The Center of the State of Texas

BACKGROUND: Review center of mass and numerical integration. Recall that the center of mass of a region bounded above by $y = f(x)$ and below by $y = g(x)$ for $a \leq x \leq b$ is the point (x_0, y_0) where

$$x_0 = \frac{1}{A} \int_a^b x\,[f(x) - g(x)]\;dx \quad \text{and} \quad y_0 = \frac{1}{A} \int_a^b \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right)\,dx$$

and A represents the area of the region. In the problem given below, there are no analytical formulas for f and g, just numerical data. So the integrals must be computed numerically.

ASSIGNMENT: The goal of this project is to approximate the center of mass of the state of Texas from data that represent the state's boundary. The northern and southern boundaries are given by

```
> north:=[[0,0], [1,0], [2,0], [3,0], [3,4.5], [4,4.5],
> [5,4.5], [6,4.5],[6,2.2], [7,2.1], [8,1.8], [9,1.9],
> [10,1.8], [11,1.7], [11,-2.2]];
> south:=[[0,0], [1,-1.1], [2,-2.5], [3,-2.9], [4,-2.3],
> [5,-2.8], [6,-4.4], [7,-5.8], [8,-6.1], [9,-3.3],
> [10,-2.8], [11,-2.2]];
```

Here, the origin is the western corner of Texas (near El Paso) and the x-axis is the extension of the east-west border between New Mexico and Texas. Each unit represents approximately 69 miles.

NOTE: There are two y-values given in the northern boundary for both $x = 3$ and $x = 6$ (because $x = 3$ and $x = 6$ represent the two north-south boundaries of the Panhandle). Use these data to answer these questions.

1. Plot Texas. (See Exercise 2 of Chapter 3.)

2. Compute the area of Texas using the trapezoid rule to compute the integrals. (See Exercises 10 and 11 of Chapter 6.)

3. Compute the center of mass of Texas using the trapezoid rule.
   NOTE: In the formulas for center of mass, the function f is approximated by (the y-components in) the `north` data, while g is approximated by the `south` data,

4. Plot Texas and its center of mass. (See Section 3.1 to see how to combine plots using `display`.)

12.16   Center of Gravity of a Parabolic Plate

BACKGROUND: Know how to compute the normal line to a curve and the center of mass of a region. Recall that the normal line to the curve $y = f(x)$ at the point $(c, f(c))$ has slope $m = \dfrac{-1}{f'(c)}$. Also the center of mass of a region bounded above by $y = g(x)$ and below by $y = f(x)$ for $a \leq x \leq b$ is the point (x_0, y_0) where

$$x_0 = \frac{1}{A} \int_a^b x\, [g(x) - f(x)]\ dx \quad \text{and} \quad y_0 = \frac{1}{A} \int_a^b \frac{1}{2} \left([g(x)]^2 - [f(x)]^2\right)\ dx$$

and A represents the area of the region.

ASSIGNMENT: Consider a thin metal plate with uniform density which occupies the region between the parabola $y = \dfrac{1}{30}x^2$ and the line $y = 120$ where the y-axis is vertical and the x-axis is the ground. In this position the center of gravity is located on the y-axis at $(0, 72)$. If the plate is rolled slightly along the x-axis, the plate will become "off-balance" and will roll to a new equilibrium state.
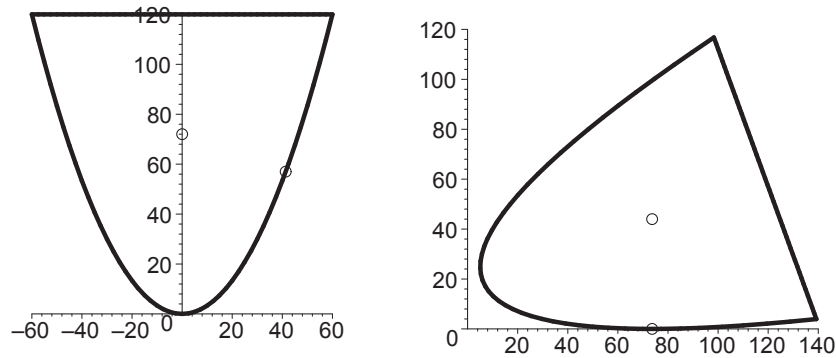
1. Verify that the original center of mass is at $(0, 72)$.

2. Give a geometric and physical argument which explains why the plate will not return to its original position and where it will roll to.

3. Find the point on the edge of the plate in the original position which will be touching the ground in the new equilibrium position.

4. Give the coordinates of the center of gravity of the plate in its new equilibrium position. Assume it rolls without slipping.

5. Repeat problems 3 and 4 for the generic parabola $y = ax^2$ where $a > 0$ (still use the line $y = 120$).
   HINT: Start by executing `assume(a>0);`

6. Give a parametric plot (with the parameter a) of the centers of gravity found in problem 5. What's the smallest allowed value of a?

12.17 The Oil Tank

BACKGROUND: Review the following applications of integration:

How to compute a volume by slicing.

How to find the work done to pump a liquid out of a tank.

How to find the center of mass of a solid.

ASSIGNMENT: The oil tank[3] below is cylindrical with hemispherical "caps". The length of the cylinder part is $L = 40\,ft$ and the radius of the cylinder (and of each hemispherical cap) is $r = 5\,ft$.
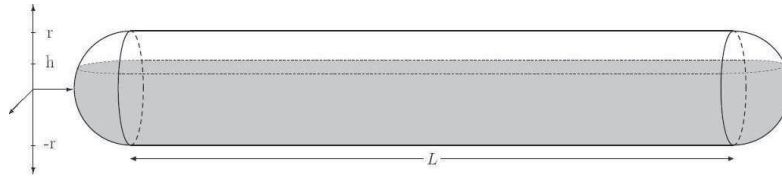
The tank contains $V = 22,300\,gal$ of Texas crude oil whose density is $\rho = 54.5\,lb/ft^3$. Find:

1. The height h of the oil above the center of the tank.

2. The weight (or force induced by gravity) of the oil.

3. The amount of work required to pump the oil out of the top of the tank.

4. The distance below the center of the tank where the center of gravity of the oil occurs.

You may assume that the acceleration of gravity is $g = 32\,ft/sec^2$ and $1ft^3 = 7.481\,gal$.

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[3]This project was originally developed by Troy Henderson, Texas A&M Univ.

12.18   The Skimpy Donut

GOAL: In this project, you will compute a volume of revolution, a surface area of revolution, and solve a max/min problem.

ASSIGNMENT: The GETFAT Donut company makes donuts with a thin layer of chocolate icing. The company decides to cut costs by minimizing the amount of chocolate icing used on each donut without shrinking the volume of the donut.
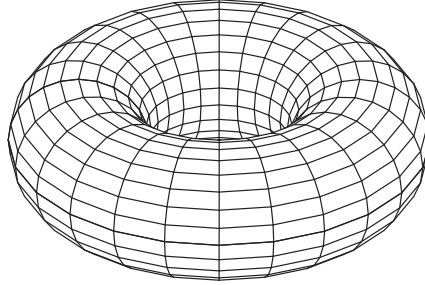
The donut has the shape of a torus which is formed by revolving the circle $(x - a)^2 + y^2 = b^2$ around the y-axis. (Here b is the radius of the circle and a is the distance from the center of the hole to the center of the circle.) At present the company makes donuts with $a = 5$ cm and $b = 3$ cm. The problem, then, is to determine the dimensions of the donut with the same volume that minimize the surface area. Follow these steps:

1. Compute the volume V of the donut as a function of a and b using the technique of either washers or cylindrical shells. As a check, verify the volume of the donut with $a = 5$ and $b = 3$ is $90\pi^2 \approx 888$ cm^3.
   HINT: First execute `assume(b>0, a>b);`

2. Compute the surface area of the donut as a function of a and b either by solving the equation of the circle for x or y or by parametrizing the circle as $x = a + b\cos t$, $\ y = b\sin t$. As a check, verify the surface area of the donut with $a = 5$ and $b = 3$ is $60\pi^2 \approx 592$ cm^2.

3. With the volume fixed at $90\pi^2$, find the dimensions a and b of the donut which minimize the surface area of the donut.
   NOTE: For this problem, you will have to determine the range of allowable a and b.

4. Is there a maximum surface area for the given volume? (So the company could advertise extra chocolate.)

5. Write a report to the CEO summarizing your recommendations (including the percent savings or percent extra cost). Anything you say in this report must be documented in an appendix of Maple computations for the engineers.

12.19   Area Between a Curve and Its Tangent Line

ASSIGNMENT: In this project,[4] you will find the tangent line to the graph of a function for which the area between the curve and the tangent line is a minimum.

1. Pick a function $y = f(x)$ which is everywhere concave up or everywhere concave down, such as $y = f(x) = -x^2$. NOTE: If the concavity changes, then the tangent line might cross the curve, which we don't want.

2. Find its tangent line at a general point $x = p$.

3. Compute the area between the curve and its tangent line at $x = p$ above the interval $0 \le x \le 1$. Label it `Area`.

4. Find the point $x = p_{\min}$ for which `Area` is a minimum. Be sure to apply the Second Derivative Test to verify that your critical point is a minimum.

5. Plot the curve and the tangent line for several values of p in $[0, 1]$ including the minimum. Plot the `Area` function.

6. Repeat steps 1-5 for three or more other functions $f(x)$. Use interesting functions, not just polynomials, and check the concavity on the interval $[0, 1]$. Be sure to try functions which are concave up as well as concave down.

7. What do you conjecture?

8. Prove your conjecture by repeating steps 1-4 for an undefined function `f:=g(x)`, once assuming g is concave up and once assuming g is concave down. Before solving for p you will need to give names to the derivatives of g using, for example, `subs(diff(g(p),p,p)=ddg, ...)`.

9. What happens to your conjecture and proof if you change the interval from $[0, 1]$ to $[a, b]$?

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[4]The idea for this project was originally suggested by Carol Scheftic, Cal. Poly. St. Univ.

12.20   Curves Generated by Rolling Circles

BACKGROUND: Review parametric curves and their slope and arc length.

ASSIGNMENT: In this project, you will study the cycloid, the epicycloid and the hypocycloid, which are curves generated by a point on a rolling circle.

1. Consider a wheel of radius R. Fix a point P on the rim of the wheel, initially at the bottom. Now let the wheel roll on level ground and consider the path traced out by the point P. (See the figure.) This path is called a *cycloid*.
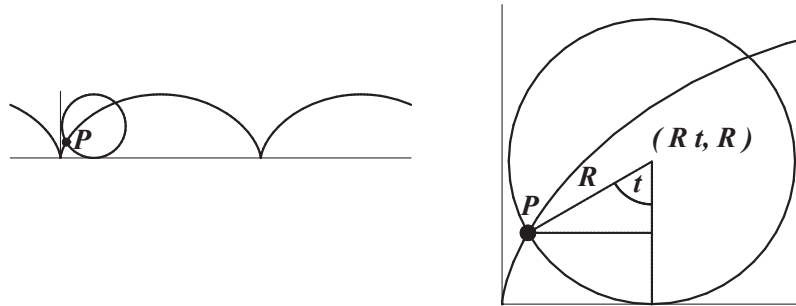
(a) Show that the cycloid is parameterized by the formulas

$$x(t) = R\left(t - \sin(t)\right) \qquad \text{and} \qquad y(t) = R\left(1 - \cos(t)\right)$$

Here, θ is the angle between the vertical and the ray that extends from the center of the circle to P (so $\theta = 0$ when P is at the origin). Enter them into Maple as: `x1,y1:=R*(t-sin(t)),R*(1-cos(t));`

(b) Plot two arches of this cycloid with $R = 1$. To plot a parametric curve (x_1, y_1) (where x_1 and y_1 are expressions in t) over the interval $a \le t \le b$, use the command: `plot([x1,y1,t=a..b]);`
NOTE: You must use square brackets with this plot command. Now unassign R using `R := 'R';`

(c) The arc length of a parametric curve $(x(t), y(t))$ for $a \le t \le b$ is

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

Compute the arc length of one arch of this cycloid for a general value of R. (If necessary, `assume` R is positive. See `?assume`.)

(d) The slope of a parametric curve $(x(t), y(t))$ is given by $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$.
Find the slope of the cycloid at $t = \dfrac{\pi}{3}$.

(e) Find the slope of the cycloid as $t \to 0^+$ by computing an appropriate limit. (See `?limit[dir]` to learn about one-sided limits.)

2. Now suppose a circle of radius a rolls around the outside of a circle of radius $R > a$ centered at the origin. (See the figure.) The path of a fixed point P on the rolling circle is called an *epicycloid*.
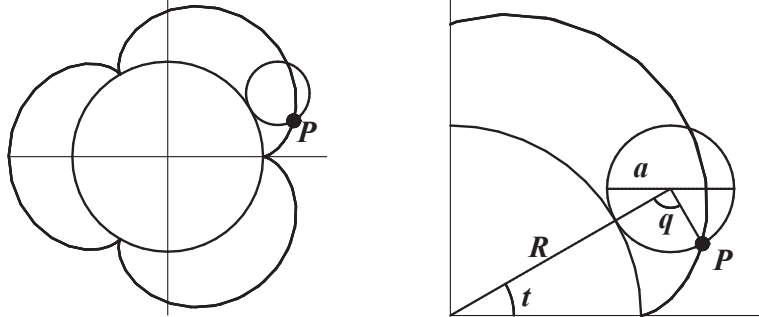
(a) Find the parameterization the epicycloid. Here t is the angle measured counterclockwise from the positive x-axis to the line segment that runs from the origin to the center of the rolling circle. Assume that P is located at the point $(R, 0)$ when $t = 0$.
    HINT: Show that the angle q in the figure is $q = Rt/a$.

(b) Plot the epicycloid with $R = 3$ and $a = 1$.

(c) Compute the arc length of one of the arches of the epicycloid. (Assume R is positive, a is positive and $R > a$.)

3. Now suppose a circle of radius a rolls around the inside of a circle of radius $R > a$ centered at the origin. (See the figure.) The path of a fixed point P on the rolling circle is called an *hypocycloid*.
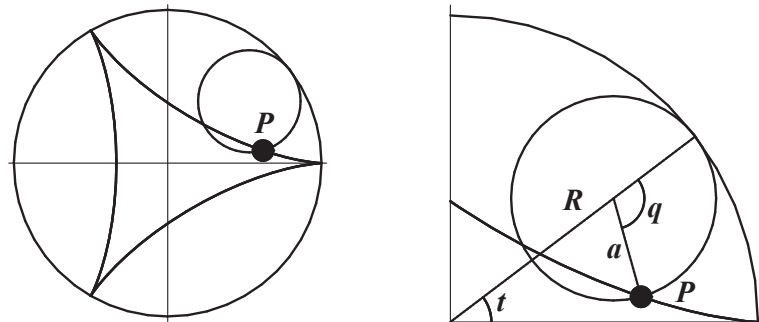
(a) Find the parameterization the hypocycloid. Here t is the angle measured counterclockwise from the positive x-axis to the line segment that runs from the origin to the center of the rolling circle. Assume that P is located at the point $(R, 0)$ when $t = 0$.
    HINT: Show that the angle q in the figure is $q = Rt/a$.

(b) Plot the hypocycloid with $R = 3$ and $a = 1$.

(c) Compute the arc length of one of the arches of the hypocycloid. (Assume R is positive, a is positive and $R > a$.)

12.21   The Wankel Rotary Engine

BACKGROUND: You need to know how to plot polygons and parametric curves, how to compute the area between two curves and how to use the `piecewise` and `animate` commands. In particular the area under a parametric curve $(x(t), y(t))$ is

$$A = \int_{x_1}^{x_2} y(x)\, dx = \int_{t_1}^{t_2} y(t) \frac{dx}{dt}\, dt$$

Further the length of an arc of a circle with central angle θ is $s = r\theta$.

GOAL: In this project you will study the motion of Felix Wankel's rotary engine[5] of the type once used by Mazda. The basic geometry consists of two circles and an equilateral triangle which just fits inside a curve called an epitrochoid.
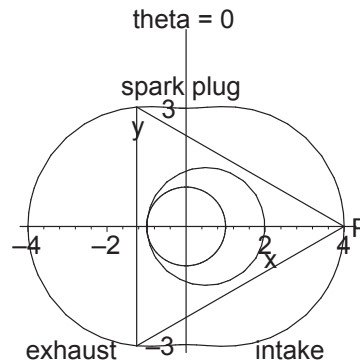

In our simple model, the inner circle has radius 1, center at the origin and does not move. The outer circle has radius $3/2$, is always tangent to the inner circle and is constrained to roll counterclockwise without slippage around the inner circle. The equilateral triangle is concentric with the outer circle and has vertices which are at distance $7/2$ from the center of the triangle. This triangle, called the rotor, rolls with the outer circle. At time $t = 0$ the two circles are tangent to each other at the point $x = -1$ while one vertex of the triangle (called P) is on the positive x-axis at $x = 4$ as shown above. As time progresses, the outer circle rolls around the inner circle and the vertex P traces out a parametric curve $P(\theta) = (x(\theta), y(\theta))$ as a function of the angle θ through which the point of tangency has moved on the inner circle, measured counterclockwise from the negative x-axis. This curve is called an epitrochoid, and is the shape of a cross section of the "cylinder" of a Wankel engine.

In the engine, each of the three spaces between the rotor and the walls of the cylinder is called a "chamber". As the rotor revolves, each chamber passes through the various phases of the Otto cycle: injection, compression, ignition, expansion and exhaust. We will describe these phases for the chamber in front of the vertex P:

[5]For more information, see the Wikipedia entry on the Wankel engine.

From $\theta = -5\pi/2$ to $\theta = -\pi$, the fuel-air mixture is injected through the open intake valve on the lower right side:
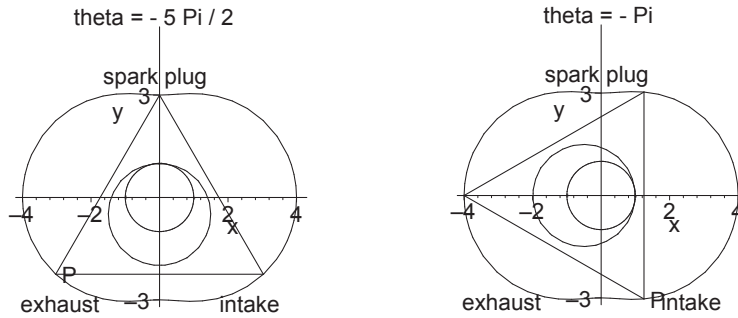
From $\theta = -\pi$ to $\theta = \pi/2$, the fuel-air mixture is compressed.

Just after maximum compression at $\theta = \pi/2$, the spark plug ignites the fuel.
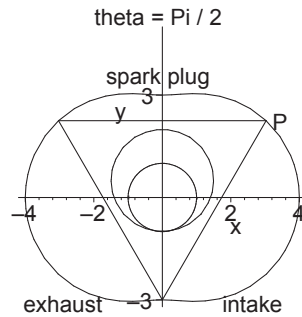
From $\theta = \pi/2$ to $\theta = 2\pi$, the fuel expands providing a power stroke.

From $\theta = 2\pi$ to $\theta = 7\pi/2 \approx -5\pi/2$, the used fuel is compressed and escapes through the open exhaust valve on the lower left side.
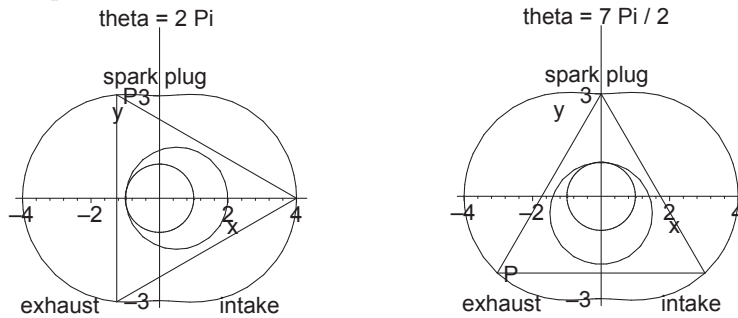
ASSIGNMENT:

1. The inner circle does not move. Write an equation, `eq1`, for the inner circle and use `implicitplot` to plot it.

2. The center of the outer circle (which is also the center of the triangular rotor) changes with the angle to the point of tangency with the inner circle, θ. Write an equation, `eq2`, for the outer circle for general θ. Then use `subs(theta=Pi/3,eq2)` and `implicitplot` to plot it when $\theta = \pi/3$.

3. Explain why, if the point of tangency of the two circles has moved coun-
terclockwise around the small circle through θ radians, then the vertex P
which was originally on the x-axis will be at

$$x_1 = \frac{1}{2}\cos(\theta) + \frac{7}{2}\cos\left(\frac{\theta}{3}\right) \qquad \text{and} \qquad y_1 = \frac{1}{2}\sin(\theta) + \frac{7}{2}\sin\left(\frac{\theta}{3}\right).$$

HINTS: The main two parts of the derivation are showing (i) that the
center C of the triangle rotates through an angle θ and (ii) that the vertex
P rotates through an angle $\theta/3$ around the center C. Below is a diagram
which shows the inner and outer circles after the point of tangency has
rotated by θ radians.
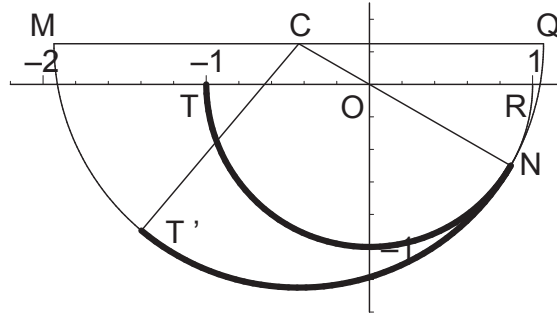
In the diagram, the center of the (small) stationary circle is at the origin,
O. The center of the (large) rotor circle (and the triangle) is at C. The
two circles are currently tangent at N. The points T and T' are the
points on the inner and outer circles which used to be at the initial point
of tangency and the thickened arcs represent the parts of the two circles
that have been in contact with each other. Notice that N is diametrically
opposed to C and that T' is diametrically opposed to the vertex P of the
triangle (not shown). In other words, N, O and C are colinear and P, C
and T' are colinear. So you need to show $\angle ROC = \theta$ and $\angle MCT' = \theta/3$.
Given that $\angle TON = \theta$, identify each of the following and justify your
answer: (When you answer (b) and (d), you will need to use the arclength
formula $s = r\theta$ where θ is in radians, $r = 1$ for the small circle and $r = 3/2$
for the large circle.)

(a) the angle $\angle ROC$ (which completes part i)

(b) the arclength $\overset{\frown}{TN}$

(c) the arclength $\overset{\frown}{T'N}$

(d) the angle $\angle T'CN$

(e) the angle $\angle NOR$

(f) the angle $\angle NCQ$

(g) the angle $\angle MCT'$ (which completes part ii)

(h) the coordinates of the point C.

(i) the coordinates of the point P.

4. Use the previous result to do a parametric plot of the (epitrochoid) cylinder wall along which the vertex P of the triangular rotor travels.

5. The position of the vertex P of the rotor is given in #3 above. The other two vertices of the triangular rotor are $2\pi/3$ apart. Write down the positions of the other two rotors as (x_2, y_2) and (x_3, y_3) for general θ. Then use `subs` and `plot` to plot the triangle when $\theta = \pi/3$.

6. Combine the plots of both circles, the triangle and the epitrochoid for $\theta = \pi/3$ using `display`.

7. Write a Maple procedure `F(theta)` which plots both circles, the triangle and the epitrochoid at the same time as a function of θ. (See Section 10.3.)

8. Make the outer circle and the triangle roll around the inner circle by animating your plot. (See `?animate`.) To see it move, you need to right click in the plot and select ANIMATE > PLAY or click in the plot and then click on the PLAY button which is a triangle pointing to the right on the toolbar.

9. Write a Maple procedure with argument θ to compute the area between a side of the rotor and the epitrochoid. Observe that for many angles the top or bottom curve will be split into two parts: an edge of the rotor and a piece of the epitrochoid. Use Maple's `piecewise` command to make sure that the correct rules are applied at the correct angles. Look at your animation to determine the angles at which the changes occur.

10. Plot the graph of the area between the rotor and the wall over the interval $(0, 3\pi)$. From the plot, what are the values of θ (expressed as multiples of π) which give the minimum and maximum area? Using your area function, what are the minimum and maximum area?

11. The difference between the maximum and minimum area is known as the *displacement* of the engine. The ratio of the maximum area to the minimum area is called the *compression ratio*. Compute the displacement and compression ratio for your Wankel engine. (Higher compression ratios can be achieved by "rounding" the rotor in an appropriate manner, but we will not deal with that here.)

12.22   Shakespeare's Shylock

BACKGROUND: This lab introduces compound interest and continuous compounding. In the process, you will discover an important limit. You will need the Maple commands for functions of several variables (See Section 2.1.) and computing limits at infinity. (See Sections 5.3 and 9.1.)

ASSIGNMENT: Shakespeare's Shylock used to lend money at the usurious rate of 100% per year simple interest. Thus if you borrow $1000 (and don't pay any back) then at the end of 1 year you owe $2000, and at the end of 2 years you owe $3000 dollars. Being financially savy, Shylock realized he could make more money if he compounded the interest annually, or more frequently or even continuously. Let's investigate what he discovered. We first have some preliminary questions, and then we will come back to Shylock.

1. Suppose you put $1000 in a bank at 10% compounded annually for 2 years and don't make any withdrawls. (Use Maple as a calculator.)

    (a) How much interest will you receive at the end of the first year?

    (b) How much money will you have in the bank at the end of 1 year?

    (c) For the second year, the money you have at the beginning of the second year will receive 10% interest. How much interest will you receive at the end of the second year?

    (d) How much money will you have in the bank at the end of 2 years?

2. Suppose you put P (principal) in a bank which pays $100r\%$ interest compounded annually. (If the rate is 5%, then $r = .05$.)

    (a) In terms of P and r, how much will be the bank at the end of 1 year?

    (b) After 2 years?

    (c) After t years? Why?
    HINT: Factor your answer to part (b) using the `factor` command.
    ANSWER: $A = P(1 + r)^t$
    Each year, the amount in the bank is multiplied by a factor of $(1+r)$. Multiplying the previous amount by the 1 says this money is still in the bank. Multiplying by the r gives the interest for the year. Adding these together gives the principal at the end of the year.

    (d) Express your answer as a Maple function `A` of the three variables `P`, `r` and `t`, as follows

        > A := (P,r,t) -> P*(1 + r)^t;

    (e) Use your Maple function `A` to compute the amount in the bank if $3000 is invested for 5 years at 6% interest compounded annually.

3. Suppose you put $1000 in a bank which pays 10% interest compounded semi-annually (i.e. twice a year you receive 5% interest). (Again, use Maple as a calculator.)

    (a) How much is in the bank after 6 months?

(b) After 1 year?

(c) After 2 years?

4. Suppose you put $\$P$ in a bank which pays $100r\%$ interest compounded n times a year (i.e. n times a year you receive $\dfrac{100r}{n}\%$ interest).

(a) In terms of P, r and n, how much money will you have in the bank at the end of 1 year?

(b) After 2 years?

(c) After t years? Why?

(d) Express your answer as a Maple function A of the four variables P, r, n and t.

(e) Use your Maple function A to compute the amount in the bank if $3000 is invested for 5 years at 6% interest compounded monthly.

5. Now back to Shylock. Suppose you borrow $1000 at 100% interest ($r = 1.00$) compounded n times a year. How much money will you owe after 1 year, if you don't pay anything back and it is compounded

(a) semi-annually? (Use your formula from 4(d).)

(b) monthly?

(c) weekly? (Assume 52 weeks in a year.)

(d) daily? (Assume 365 days in a year.)
HINT: At this point, execute Digits := 20; and be sure to put a decimal point in the rate: $r = 1.00$.

(e) every minute?

(f) every second?

(g) continuously?
HINT: Use Maple's Limit and value commands. You will be able to derive this limit once you learn l'Hospital's rule.

6. Write the answer to 5(g) in terms of known mathematical constants.

7. Now let's be realistic. Suppose you put $1000 in a bank at 5% interest compounded continuously.

(a) How much would you have at the end of 1 year? (Use your formula from 4(d) and Maple's Limit and value commands.)

(b) Compute the number $e^{.05}$ and compare it to your answer to (a).
NOTE: In Maple, this is entered as exp(.05).

8. Suppose you put $\$P$ in a bank at $100r\%$ interest compounded continuously.

(a) How much money you will you have in the bank after 1 year? (Use your formula from 4(d) and Maple's Limit and value commands.)

(b) How much money will you have in the bank after t years? Justify the answer.

12.23   The Bouncing Ball

BACKGROUND: Review geometric series and Newton's Laws.

ASSIGNMENT: A ball is dropped from a height of 54 feet. Each time it bounces it reaches a height which is 2/3 of the height on the previous bounce. Thus after the first bounce it reaches 36 feet, then 24 feet, then 16 feet, etc.

1. What is the total distance travelled by the ball (after an infinite number of bounces)? HINT: Use `Sum` and `value`.

2. What is the total time the ball takes to travel this distance? Assume Newton's Law of gravity with no air resistance. Thus, the distance h the ball falls is related to the time t by $h = \dfrac{1}{2}gt^2 = 16t^2$.
   HINT: You may need the `rationalize` command.

12.24   Pension Funds

BACKGROUND: Review separable differential equations and exponential growth.

ASSIGNMENT: A pension fund starts out with $\$P$ (at $t = 0$) and is invested with a return of $100r\%$ per year, compounded continuously. (Here, r is the interest rate, given as a number between 0 and 1). The pension fund must continuously pay out money at the rate of $\$R$ per year to its employees for a period of n years. (This means that the value of the pension fund decreases to zero after n years.) Let $y(t)$ denote the value of the pension fund after t years.

1. From the information given, derive the differential equation $\dfrac{dy}{dt} = ry - R$, and the conditions $y(0) = P,\ \ y(n) = 0$.

2. Solve this differential equation with the initial condition $y(0) = P$ for $y(t)$ by using `dsolve`.

3. Using the condition $y(n) = 0$, find a formula for P in terms of r, n and R. $\$P$ represents the amount of money required to pay out $\$R$ per year for n years, assuming the rate of return on the investment is $100r\%$.

4. Calculate P for $R = 40,000,\ \ r = .05$, and $n = 15$.

5. Calculate the interest rate ($100r\%$) required so that an initial value of $\$P = \$400,000$ for the pension fund will pay out $\$R = \$40,000$ per year for $n = 15$ years.

12.25 The Flight of a Baseball

BACKGROUND: Review motion in two dimensions, including velocity, accelera-
tion and Newton's Laws.

ASSIGNMENT: Imagine that a baseball player is up at home plate and hits the
ball into the outfield. What parametric equations describe the position of the
ball t seconds after it is hit? How far will the ball travel? How fast does the
ball need to be hit in order for the ball to clear the home run fence? This lab
is designed to answer these questions.

1. First consider a simplified model that ignores air resistance. In this case,
   after the ball is hit, the only force acting on the ball is the vertical force
   due to gravity. Therefore, Newton's Laws say the x- and y-components of
   the acceleration of the ball are

   $$\frac{d^2x}{dt^2} = 0 \qquad \text{and} \qquad \frac{d^2y}{dt^2} = -g$$

   where $g = 32$ ft/sec^2 is the acceleration of gravity. Integrating these
   equations with respect to t gives

   $$\frac{dx}{dt} = A \qquad \text{and} \qquad \frac{dy}{dt} = -gt + B$$

   where A and B are constants of integration. Setting $t = 0$ in the above
   equations, says A and B are the x- and y-components of the initial velocity
   of the ball, respectively. Suppose the initial speed of the ball is v_0 (in units
   of feet per second) and the initial angle of inclination of the ball is θ. Then
   $A = v_0 \cos(\theta)$ and $B = v_0 \sin(\theta)$ and the equations become

   $$\frac{dx}{dt} = v_0 \cos(\theta) \qquad \text{and} \qquad \frac{dy}{dt} = -gt + v_0 \sin(\theta)$$

   One more integration with respect to t yields

   $$x = v_0 t \cos(\theta) + C \qquad \text{and} \qquad y = \frac{-1}{2}gt^2 + v_0 t \sin(\theta) + D$$

   where C and D are constants of integration. Setting $t = 0$ and assuming
   home plate is at origin ($x = 0, y = 0$) and the batter hits the ball at height
   h ft, then $C = 0$ and $D = h$. The final parameterization of the baseball's
   path is given by

   $$x = v_0 t \cos(\theta) \qquad \text{and} \qquad y = \frac{-1}{2}gt^2 + v_0 t \sin(\theta) + h$$

   Enter these formulas into Maple as expressions `x1` and `y1` with $g = 32$
   ft/sec^2, $\theta = \pi/4$ rad, $v_0 = 125$ ft/sec and $h = 5$ ft.

2. Plot the trajectory of the ball until the ball hits level ground, using a
   parametric `plot`. Start the parameter t at 0 and solve the equation $y = 0$
   to find the time the ball hits the ground. Find the horizontal distance
   traveled by the ball.

3. What is the shape of the graph? Eliminate t, and determine the trajectory by giving y as a function of x.

   HINT: Solve the equation x=x1 for t and substitute for t in y1 using the subs command.

   NOTE: Now execute restart; and re-input the expressions x1 and y1 into Maple but with the initial angle θ, the initial speed v_0 and the time t as free variables. Keep $g = 32$ ft/sec^2 and $h = 5$ ft as constants.

4. Suppose the home run fence is 12 feet high and 350 feet from home plate. What is the minimum velocity at which the ball must leave the bat so that the ball barely clears the home run fence?

   HINT: *Be careful:* Do not assume any particular value of the angle θ. In fact your strategy should be as follows. First, eliminate t as you did in part 3 (but keeping v_0 and θ as free variables). Then substitute $x = 350$ and $y = 12$ into the resulting equation and solve for the speed v_0 in terms of the angle θ. (You may need a plot to determine which solution for v_0 is positive.) Finally, minimize v_0 as a function of θ.

5. Now assume air resistance acts on the ball. Air resistance acts in the opposite direction to the velocity of the ball and its magnitude is proportional to the speed. Newton's Laws lead to the equations

$$\frac{d^2x}{dt^2} = -k\frac{dx}{dt} \qquad \text{and} \qquad \frac{d^2y}{dt^2} = -g - k\frac{dy}{dt}$$

   Here, k is a positive friction constant, which will be given later. Execute restart and solve these equations for x and y (using dsolve with initial conditions).

6. To check for consistency, take the limit of your solution as $k \to 0$ and see if your result agrees with the solution for x and y without air resistance.

7. Repeat parts 2 and 4, taking into account air resistance with $k = 0.1$. Compare your plots and your answers to your results without air resistance.

   NOTE: In part 4, the formula for v_0 as a function of θ is ugly (involving Lambert W functions) and its derivative is uglier. You would not want to find these by hand.

Related Activities: See Stewart's Applied Projects: *Which is Faster, Going Up or Coming Down?* and *Calculus and Baseball.*

12.26 Parachuting

BACKGROUND: Review systems of differential equations with initial conditions from Section 8.4. Recall the motion of an object falling with air resistance can be found from Newton's law, *Mass × Acceleration = Force* where the force is the sum of the downward force of gravity and the upward "drag" force of the air resistance. In symbols, this equation is

$$m\frac{d^2y}{dt^2} = -mg - k\frac{dy}{dt}$$

Here, m is the mass, y is the altitude (up is positive), $g = 9.8\frac{\text{m}}{\text{sec}^2}$ is the acceleration of gravity and k is the is the (positive) drag coefficient. Notice that the velocity $v = \frac{dy}{dt}$ is negative (since the object is falling). So the drag term, $-kv$, is positive (or up) thus tending to slow down the fall. You should separate the second order differential equation into a system of two first order equations:

$$\frac{dy}{dt} = v \qquad \text{and} \qquad m\frac{dv}{dt} = -mg - kv$$

ASSIGNMENT: A sky-diver, weighing 75 kg., jumps from a plane at an altitude of 3000 meters and free falls for T_1 seconds before pulling the rip chord. The parachute takes 4 seconds to open and then the sky-diver falls with a fully open parachute. Find the maximum time, T_1, the sky-diver can free fall before pulling the rip chord and still have a "gentle" landing. While the parachute is opening assume the drag coefficient is $k = 25$kg/sec. When the parachute is fully opened, assume $k = 110$kg/sec.

A landing is defined to be "gentle" if the velocity on impact is less than the impact velocity of an object dropped (free-fall) from a height of 4 meters.

0. Find the maximum velocity `Vgentle` for a gentle landing.

1. Find the altitude `Y1` and velocity `V1` at time `T1` after the first stage of free-fall. These are the initial conditions for the second stage.

2. Find the altitude `Y2` and velocity `V2` at time `T2` = `T1+4` after the second stage when the parachute is opening. These are the initial conditions for the third stage.

3. Find the time `T3` when the sky diver hits the ground and the velocity `V3` at this time. `V3` should be a function of `T1` which you can then equate to the gentle landing velocity and solve for the maximal safe `T1`. Plot the altitude and velocity as functions of time.

4. What is the landing velocity if the sky diver freefalls for a second less or more than the maximal safe time. Comments? It may be helpful to plot `V3` as a function of `T1` to see if there is a terminal velocity.

12.27   Radioactive Waste at a Nuclear Power Plant

BACKGROUND: Review first order, linear differential equations.

ASSIGNMENT: A nuclear power plant produces a waste product that is a radioactive isotope, called A. The isotope A has a half-life of 8 years and splits into radioactive isotopes B and C. 45% of the weight of A becomes B and 55% becomes C. Thus, for example, if 100 kilograms of A decay there will be 45 kilograms of B and 55 kilograms of C. Isotope B has a half-life of 15 years while isotope C has a half-life of 20 years and they decay into nonradioactive by-products. Answer the following questions.

1. Find the three decay constants.

2. Suppose you start with 300 kilograms of isotope A. What is the maximum amount of isotope B that will be present, and when will this occur? Answer the same question for isotope C.

3. When the power plant was first turned on, there was no isotope A, B, or C present. If the power plant operates so that it produces isotope A at the constant rate of 30 kilograms per year, what are the largest amounts of isotopes A, B and C that will be present, and when will these occur?

4. Federal safety requirements say that the reactor can never have on hand more than 600 kilograms of isotope A, 500 kilograms of isotope B, or 800 kilograms of isotope C. What is the maximum rate at which the power plant can produce isotope A without violating the federal regulations?

HINT 1:   If a radioactive isotope is being produced by some source at the same time as it is decaying, how does that alter the differential equation for the rate of change of the amount of this isotope?

HINT 2:   Be sure to plot A, B, and C, in order to ascertain if they have the qualitative behavior you expect.

HINT 3:   In problem 4, repeat the computations of problem 3, but assume the power plant produces isotope A at the constant rate of R kilograms per year.

12.28 Visualizing Euler's Method

BACKGROUND: Review tangent lines and differential equations. Euler's method, constructs a numerical solution $y = F(x)$ on an interval $a \leq x \leq b$ of a differential equations of the form:

$$\frac{dy}{dx} = f(x, y) \quad \text{with the initial condition} \quad y(a) = y_0$$

NOTE: If $f(x, y)$ is independent of y, then the differential equation becomes $y' = f(x)$ and the solution is just the antiderivative $y = y_0 + \int_a^x f(x)\, dx$. So Euler's method provides a numerical antiderivative.

Euler's method starts by dividing the interval into n subintervals (called steps) each of width $h = \dfrac{b-a}{n}$ (called the step size) and naming the points $x_i = a + ih$, for $i = 0, 1, 2, ..., n$. Thus $x_0 = a$ and $x_n = b$. At each point $x = x_i$, the tangent line to $y = F(x)$ is $y = F(x_i) + F'(x_i)(x - x_i)$. However, from the differential equation, $F'(x_i) = \dfrac{dy}{dx} = f(x_i, F(x_i))$. So the tangent line becomes $y = F(x_i) + f(x_i, F(x_i))(x - x_i)$. Starting with the initial condition $F(x_0) = y_0$, Euler's method recursively approximates $F(x_{i+1})$ by

$$y_{i+1} = y_i + f(x_i, y_i)\, h \qquad \text{for} \quad i = 0, 1, 2, ..., n-1$$

which is the linear approximation except that $F(x_i)$ has been replaced by y_i, found in the previous step. Euler's method can be implimented in Maple by using a `do` loop as shown in the following example. Be sure to try it!

EXAMPLE: Use Euler's method to approximate the solution to $\dfrac{dy}{dx} = y - 3x$ with $y(0) = 2$ on the interval $[0, 4]$ using 8 intervals. Plot the direction field, the exact solution and Euler's approximation in the same plot.

SOLUTION: Define the function, the interval, the number of steps and the initial condition. Then compute the step size.

```
>   f:=(x,y)->y-3*x;
>   a,b:=0,4; n:=8;
>   x[0], y[0]:=0, 2;
>   h:=(b-a)/n;
```

Now apply Euler's method:

```
>   for i from 0 to (n-1) do
>   x[i+1]:=x[i]+h;  y[i+1]:=evalf(y[i]+f(x[i],y[i])*h);
>   end do;
```

The Euler approximation is plotted using a point `plot`:

```
>   plotlist:=[seq([x[i],y[i]],i=0..n)];
>   p1:=plot(plotlist, 0..4, -10..10, thickness=3):  p1;
```

The direction field and the exact solution are plotted with the `DEplot` command in the `DEtools` package and the graphs are combined using the `display` command in the `plots` package. We need the packages, the differential equation and the initial condition.

```
>  with(plots):  with(DEtools):
>  deq:=diff(Y(X),X)=f(X,Y(X));
>  init:=[[0,2]];
```

NOTE: We have to use capital X and Y since x and y are already used for the Euler approximation.

```
>  p2:=DEplot(deq,Y(X), X=0..4, Y=-10..10, init):  p2;
>  display(p1,p2);
```

In this plot, notice that each straight line segment in the Euler approximation has a slope which agrees with the direction field at the *left* endpoint of the line segment. Also notice that the Euler approximation is above (an over estimate for) the true solution. You will be asked to explain these facts in the exercises.

ASSIGNMENT:

1. Use Euler's method to approximate the solution to $\dfrac{dy}{dx} = \dfrac{x}{y+4}$ with $y(-2) = 1$ on the interval $[-2, 4]$ using 3 intervals.

2. Find the exact solution using dsolve. (Since the equation is separable, you should be able to check this by hand.) Plot the direction field, the exact solution and Euler's approximation in the same plot.

3. Explain why each straight line segment in the Euler approximation has a slope which agrees with the direction field at the *left* endpoint of the line segment.

4. Notice that here the Euler approximation is below (an under estimate for) the true solution, while in the earlier example the Euler approximation was an over estimate. Explain this fact based on whether the solution is increasing or decreasing or concave up or concave down.

5. Repeat Exercises 1 and 2 but with $n = 12$. What happened to the graph of the Euler approximation?

6. Euler's method is usually too inaccurate to be of practical use. A significant improvement is the following, called the modified Euler's method (or Huen's method, or second order Runge-Kutta)

$$y_{i+1} = y_i + \frac{k_1 + k_2}{2}\, h \quad \text{where } k_1 = f(x_i, y_i) \text{ and } k_2 = f(x_i + h, y_i + hk_1)$$

which averages the slopes at x_i and x_{i+1}. Write a do loop to implement this method. Apply it to the Example and to Exercises 1 and 2 using $n = 3$ and $n = 12$. Plot the exact solution together with the Euler approximation and the Runge-Kutta approximation.

12.29 The Brightest Phase of Venus

BACKGROUND: Study trigonometry, polar and spherical coordinates and max/min problems and area. (This is the hardest project.)

ASSIGNMENT: The brightness of Venus is proportional to the area of the visible illuminated portion of Venus and inversely proportional to the square of the distance from the Earth to Venus. From the figure, note that, as the angle α increases from 0 to π, the area of the visible illuminated portion of Venus increases, which tends to increase the brightness of Venus. However, the distance d from the Earth to Venus also increases, which tends to decrease the brightness of Venus. For some angle α, between 0 and π, Venus will appear brightest. Find this brightest phase of Venus (i.e., the angle α).
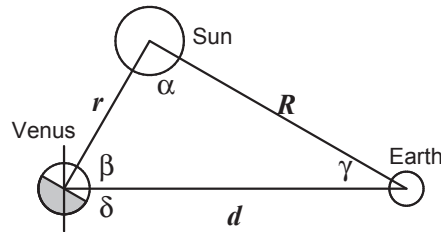
Here, $r = 67$ is the distance from the Sun to Venus and $R = 93$ is the distance from the Sun to the Earth, where the unit of distance is a million miles. As mentioned above, the brightness is

$$B = k \, \frac{\text{Area of the visible illuminated portion of Venus}}{d^2}$$

where k is a proportionality constant. We want to express B in terms of α. Here are some suggestions that may help.

1. The distance d can be found from the law of cosines:

$$d^2 = r^2 + R^2 - 2rR\cos(\alpha)$$

2. Choose a coordinate system with Venus at the origin, the Earth on the positive x-axis and the Sun in the first quadrant in the xy-plane. Then the z-axis points out of the paper. In terms of the polar (or spherical) coordinate θ, the sun illuminates from $\theta = -\delta$ to $\theta = \pi - \delta$. (See the figure.) However, the earth can only see the portion with $-\delta \leq \theta \leq \pi/2$. Venus is actually a sphere (of radius a), but from the Earth we see the disk which is the projection of the sphere into the yz-plane. The area we actually want is the illuminated portion of this disk. In spherical coordinates, the surface of Venus is

$$\left.\begin{array}{l} x = a\sin\phi\cos\theta \\ y = a\sin\phi\sin\theta \\ z = a\cos\phi \end{array}\right\} \qquad \text{for} \qquad \left\{\begin{array}{l} 0 \leq \phi \leq \pi \\ -\pi \leq \theta \leq \pi \end{array}\right.$$

So the projection of the illuminated portion on the yz-plane is

$$\left.\begin{matrix} y = a\sin\phi\sin\theta \\ z = a\cos\phi \end{matrix}\right\} \qquad \text{for} \qquad \left\{\begin{matrix} 0 \le \phi \le \pi \\ -\delta \le \theta \le \pi/2 \end{matrix}\right.$$

The edges of this illuminated projection are two parametric curves:
When $\theta = \pi/2$ the edge is the semicircle $y = a\sin\phi, \ z = a\cos\phi$.
When $\theta = -\delta$ the edge is the semi-ellipse $y = -a\sin\phi\sin\delta, \ z = a\cos\phi$.
Eliminate the parameter ϕ to find equations for the edges and integrate to find the area as a function of the angle δ.

3. It remains to express δ in terms of α. This can be done with the complementary angle relation $\beta + \delta = \dfrac{\pi}{2}$ and the law of sines $\dfrac{\sin\beta}{R} = \dfrac{\sin\alpha}{d}$
   NOTE: In solving for β, remember arcsin always returns an angle between $-\pi/2$ and $\pi/2$. So you need two different formulas for $\beta \le \pi/2$ and $\beta > \pi/2$. How is $\beta > \pi/2$ expressed in terms of R, r and d? In terms of α? This leads to two formulas for δ.

4. Combine these results to express the brightness B in terms of the one variable α. You again need two formulas for B, one for $\beta > \pi/2$ and one for $\beta < \pi/2$. Use Maple to maximize B over the interval $0 \le \alpha \le \pi$. From your answer, which is more important, increasing the illuminated portion of Venus or decreasing the distance from the Earth to Venus?